Schrödinger's Register: Foundational Issues and Physical Realization

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***Abstract*—This work-in-progress paper consists of four points which relate to the foundations and physical realization of quantum computing. The first point is that the qubit cannot be taken as the basic unit for quantum computing, because not every superposition of bit-strings of length n can be factored into a string of n-qubits. The second point is that the "No-cloning" theorem does not apply to the copying of one quantum register into another register, because the mathematical representation of this copying is the identity operator, which is manifestly linear. The third point is that quantum parallelism is not destroyed only by *environmental decoherence*. There are two other forms of decoherence, which we call *measurement decoherence* and *internal decoherence*, that can also destroy quantum parallelism. The fourth point is that processing the contents of a quantum register "one qubit at a time" destroys entanglement.**

***Keywords*-qubit; entanglement; decoherence; no-cloning theorem; quantum register.**

I. Introduction

This paper will make four points. Points (A) and (B) are foundational. Points (B), (C), and (D) relate to the physical realization of quantum computing. We will state the points and then elaborate on them.

A. The basic element of quantum computing is not the *qubit* but the *q-string.* The qubit is not basic because not every q-string can be factored into a string of qubits.
B. A *processing step* in quantum computing is defined as the application of a unitary linear operator on q-strings [4]. For physical realization purposes, this definition is incomplete. In a real quantum computer, q-strings of length n "live" on n-bit registers in superposed states. To specify a processing step, one must specify the input and output registers. Let ψ be a q-string. A "cloning" function f_{12} can be defined as $f(\psi$ on register 1$)$ = ψ on register 2; the function copies ψ from register 1 to register 2. The existence of such a processing step does not violate the "No-cloning" theorem.
C. The power of quantum computing depends on quantum parallelism. Quantum parallelism is destroyed if q-strings decohere during processing. Experimental results indicate that the time to environmental decoherence is inversely related to the size of the physical system considered. If this is so, then the longer the q-string, the shorter the time to its environmental decoherence. This rules out quantum parallelism for q-strings of arbitrary length. Besides environmental decoherence, two other kinds of decoherence are introduced and discussed.
D. If a q-string is processed "one qubit at a time", then the resulting q-string is a string of qubits. So, any entanglement in the original q-string is destroyed.

II. Elaborations of the Four Points

A. Qubits and Q-Strings

A *bit* = b_i is 0 or 1. A string of bits of length $n =$ $\mid b_1 \ldots b_n\rangle$. The number of all strings of bits of length $n = 2^n$. A *q-string of length n* is a sum of the bit strings of length *n* weighted by complex numbers. So, a q-string of length $n =$

$$\psi = \sum_{m=1}^{m=2^n} c_m \phi_m ,$$

where the ϕ_m are the bit strings of length n and the complex numbers c_m satisfy the condition $\sum_m \mid c_m \mid^2 = 1$. Some of the c_m may be 0, so it may be that not every bit string of length n is a nonzero-weighted component of the q-string. A *qubit* q_i = a q-string of length 1. So, $q_i = \alpha_i \mid 0\rangle + \beta_i \mid 1\rangle$, where $\mid \alpha_i \mid^2 + \mid \beta_i \mid^2 = 1$. A *string of qubits* is a product of qubits $= q_1 \otimes q_2 \otimes \ldots q_n$.

Consider a string of two qubits:

$$\begin{aligned} q_1 \quad \otimes \quad q_2 &= (\alpha_1 \mid 0\rangle + \beta_1 \mid 1\rangle) \otimes (\alpha_2 \mid 0\rangle + \beta_2 \mid 1\rangle) \\ = \quad & \alpha_1\alpha_2 \mid 00\rangle + \alpha_1\beta_2 \mid 01\rangle + \beta_1\alpha_2 \mid 10\rangle + \beta_1\beta_2 \mid 11\rangle . \end{aligned}$$

$\mid \alpha_1 \mid^2 + \mid \beta_1 \mid^2 = 1$ and $\mid \alpha_2 \mid^2 + \mid \beta_2 \mid^2 = 1$.

Now consider the q-string of length 2:

$$0 \mid 00\rangle \; + \; \frac{1}{\sqrt{2}} \mid 01\rangle \; + \; \frac{1}{\sqrt{2}} \mid 10\rangle \; + \; 0 \mid 11\rangle$$

If this q-string = $q_1 \otimes q_2$, then $\alpha_1\alpha_2 = 0$, $\alpha_1\beta_2 = \frac{1}{\sqrt{2}}$, $\beta_1\alpha_2 = \frac{1}{\sqrt{2}}$, and $\beta_1\beta_2 = 0$. So, either $\alpha_1 = 0$ or $\alpha_2 = 0$. But if $\alpha_1 = 0$, then $\alpha_1\beta_2 = 0 \neq \frac{1}{\sqrt{2}}$. If $\alpha_2 = 0$, then $\beta_1\alpha_2 = 0 \neq \frac{1}{\sqrt{2}}$. So the above q-string of length 2 is not a string of 2 qubits. This argument can be generalized. So, not every q-string of length n is a string of n qubits. The above q-string of length 2 is said to have "entangled qubits", because it cannot be factored into a string of 2 qubits.

B. *Q-Strings and the No-cloning Theorem*

The physical realization of a q-string of length n in a real quantum computer will be the state of a register with n bit positions. Suppose a real quantum computer contains two n-bit registers. Suppose register 1 contains a q-string of length n, i.e., suppose register 1 is in a superposition of states where each of the states is a bit-string of length n. Suppose register 2 contains a string of n zeroes. Nothing in what has been described so far rules out that the computer can execute the command: *Store the content of register 1 in register 2*. The result of this processing will be two registers, each containing the same q-string. Wouldn't this violate the "No-cloning theorem" [2]?

A *processing step* in quantum computing is defined as the application of a unitary linear operator f on q-strings of length n [4]. Suppose

$$\psi = \sum_{m=1}^{m=2^n} c_m \phi_m,$$

a q-string of length *n*. Since f is linear, $f(\psi) = \sum_{m=1}^{m=2^n} c_m f(\phi_m)$. Since f is unitary, the action of f on a bit string yields a bit string of the same length. So,

$$f(\phi_m) = \phi_{m\prime} = \phi_{f(m)}.$$

Thus,

$$f(\psi) = \sum_{m=1}^{m=2^n} c_m \phi_{f(m)}.$$

As far as it goes, this definition of *processing step* is correct. For physical realization purposes, however, it is incomplete. In a real quantum computer, quantum strings live on n-bit registers. So, the mathematical representation of a processing step must specify the input and output registers. Thus, the "cloning" function C must be identified as C_{12}. So, $C(\psi$ on register $1) = \psi$ on register 2, and C is simply the Identity Transformation, $I_\psi = \psi$, which is unitary and linear.

The "No-cloning theorem" does not apply here. What the No-cloning theorem states is as follows: Let ψ be a q-string of length n. Let 0 be a string of n zeroes. No-cloning result: there is no unitary linear function g on q-strings of length $2n$, such that $g(\psi \otimes 0) = g(\psi \otimes \psi)$.

C. *Quantum Parallelism and Decoherence*

What is *quantum parallelism*? Suppose ψ is a q-string and ψ has m different bit strings appearing as nonzero-weighted components. Then, quantum parallelism is the idea that one processing step on ψ is, in a sense, equivalent to m processing steps on the bit string components of ψ [2]. Since processing takes time, quantum parallelism is lost if the q-string *decoheres* during processing. What is *decoherence*? The only kind of decoherence discussed as such in quantum computing is *environmental* decoherence. We believe that there are two other forms of decoherence, *measurement* decoherence and *internal* decoherence, and that these other forms may pose obstacles for quantum parallelism as well. Let us start with environmental decoherence. Let

$$\psi = \sum_m c_m \phi_m$$

be the state of a physical system where the ϕ_m are the base states of the system and the c_m are the complex numbers satisfying the usual condition. Let E_0 represent the initial state of the environment. Environmental decoherence is the idea that after a time (decoherence time) the physical system interacts enough with the environment so that the state of the system plus environment evolves to the following:

$$\mid \psi, E_0\rangle \;\longrightarrow\; \sum_m c_m \mid \phi_m, E_m\rangle,$$

where the E_m are states of the environment that do not mutually "interfere". What the "non-interference" means practically is that the evolved state immediately collapses:

$$\sum_m c_m \mid \phi_m, E_m\rangle \;\longrightarrow\; \text{one of the } \mid \phi_m, E_m\rangle \text{ states}$$

with a probability of $\mid c_m \mid^2$. Q-strings live on the register of the quantum computer. So, Ψ above is the state of the register(s) of the computer, and E above is the state of the environment of the register(s); i.e., the rest of the computer plus the external world. So, if decoherence time is less than processing time, a q-string will collapse into one of its component bit strings, and quantum parallelism will be destroyed. Erich Joos [1] states that experimental results seem to indicate that decoherence time is related inversely to size; he even says (p. 13): "..macroscopic objects are extremely sensitive and immediately decohered." If what Joos says is true, then the longer the q-string, the shorter the time to its decoherence. This rules out quantum parallelism for q-strings of arbitrary length. Joos says (p. 14):

> ...(decoherence) represents a major obstacle for people trying to construct a quantum computer. Building a really big one may well turn out to be as difficult as detecting other Everett worlds!

Many think that detecting other Everett worlds is impossible [3]. *Measurement* decoherence can be explained as follows.

Let

$$\psi = \sum_m c_m \phi_m$$

be the state of a physical system, and suppose at t_0 (the initial time), ψ is coupled with a *measuring device* in state M_0. Let "measurement time" be the amount of time required for the measuring device to measure the physical system, i.e., the amount of time for the measuring device to evolve from M_0 to a superposition of *indicator* states M_i. The picture of the evolution is as follows:

$$\sum_m c_m \phi_m M_0 \quad \longrightarrow \quad \sum_m c_m \phi_m M_m.$$

If we make the assumption that the $M_i (i \geq 1)$ do not mutually interfere, then ψ immediately collapses:

$$\sum_m c_m \phi_m M_m \quad \longrightarrow$$
$$\text{one of the } \phi_m M_m \text{ with a probability of } \mid c_m \mid^2 .$$

Measurement decoherence (called *quantum measurement* by quantum computer scientists) is a resource of, and not an obstacle to, quantum computing if it occurs *after processing is complete*. Measuring the output q-string is the way to read information contained in that q-string. No one will intentionally apply a measuring device to a register or registers *before* processing is complete. So, how can measurement decoherence be an obstacle to quantum computing? A physical quantum computer will contain a register or registers, but will also contain other devices (for processing, etc.) besides registers. If the "innards" of a physical quantum computer exclusive of the registers *act like* a measuring device during processing, then there will be an unintentional measurement of a register or registers during processing, and quantum parallelism will be lost. Thus, it is a challenge not only to build registers that can exist in superposed states, but also to build the rest of the quantum computer so that it does not act like a measuring device on registers during processing.

The third form of decoherence is *internal* decoherence. Suppose we have a physical system in an initial state ψ_0. Suppose also that in some time interval (evolution time), the physical system evolves to a superposition of base states:

$$\psi \quad \longrightarrow \quad \sum_m c_m \phi_m.$$

If we assume that the ϕ_m do not mutually interfere, we have immediate collapse:

$$\sum_m c_m \phi_m \longrightarrow \text{one of the } \phi_m \text{with a probability of} \mid c_m \mid^2 .$$

In the standard two-slit experiment, we have evolution to:

$$\alpha \mid \text{particle travels through slit 1}\rangle$$
$$+ \beta \mid \text{particle travels through slit 2}\rangle,$$

but these two states mutually interfere, as is evidenced by the interference pattern built up on the photographic backstop as the experiment is repeated. So the standard two-slit experiment is *not* an example of internal decoherence.

We can get internal decoherence if we modify the two-slit experiment. Put a light source near slit 1, so that a particle traveling through slit 1 produces a light flash because a photon from the source bounces off the particle. Then we have evolution to:

$$\alpha \mid \text{particle travels through slit } 1 + \text{flash of light}\rangle$$
$$+ \beta \mid \text{particle travels through slit } 2 + \text{no flash of light}\rangle.$$

These two states do not mutually interfere, as is evidenced by the lack of interference pattern on the photographic backstop. (Remember that *observation* of the light flash is not necessary to destroy interference; only existence of the flash is necessary.

Another physical system that internally decoheres is Schrödinger's Cat Box, consisting of a box occupied by a radioactive source, Geiger counter, trip hammer, vial of cyanide, and live cat. The box evolves into:

$$\alpha \mid \text{dead cat, smashed vial, tripped hammer, etc.}\rangle$$
$$+ \quad \beta \mid \text{live cat, unsmashed vial, untripped hammer, etc.}\rangle$$

Based on all available observational evidence, these two states do not mutually interfere. No one has ever observed a superposition of $\alpha \mid \text{dead cat}\rangle \; + \; \beta \mid \text{live cat}\rangle$, let alone:

$$\alpha \mid \text{smashed vial}\rangle \; + \; \beta \mid \text{unsmashed vial}\rangle,$$
$$\alpha \mid \text{tripped hammer}\rangle \; + \; \beta \mid \text{untripped hammer}\rangle, \text{ etc.}$$

So, the solution to Schrödinger's Paradox is internal decoherence. We can talk of Schrödinger's Register instead of Schrödinger's Cat, and mean by this an n-bit register that can exist in a superposition of bit strings of length n such that those bit strings *do* interfere. (We want the bit strings to interfere, or else we would have a collapse to a single bit string and no quantum parallelism.) So, the challenge for quantum computer scientists is to build Schrödinger's Register. Good luck!

D. Qubits, Q-Strings, and Entanglement

Consider a q-string of length n, ψ. Suppose ψ is "entangled." Then it is not equal to a string of n qubits, but a string of n qubits can be constructed from it in the following way:

Survey the bit string components of ψ. Let m be a position from 1 to n in the bit string. Add the amplitudes for all components with a 0 in position m. Call the sum α_m. Add the amplitudes for all components with a 1 in position m. Call the sum β_m. Construct the qubit $(\alpha_m \mid 0\rangle + \beta_m \mid 1\rangle)$. Take the product of such qubits for all positions. This is the string of qubits to be constructed:

$$\bigotimes_{m=1}^{m=n} (\alpha_m \mid 0\rangle \ + \ \beta_m \mid 1\rangle) \ = \ \bigotimes_{m=1}^{m=n} q_m \ .$$

The result of processing ψ "one qubit at a time" = the product of the results of applying a processing step f on qubits to each qubit in the string constructed from (but not identified with) ψ:

$$\bigotimes_{m=1}^{m=n} f(q_m) \ = \ \bigotimes_{m=1}^{m=n} q'_m \ .$$

A string of m qubits has no entanglement among qubits. So, processing ψ "one qubit at a time" destroys entanglement.

III. Conclusion

We believe that the four points above will all be necessary to progress in understanding how to realize a quantum computer. Most fundamental would be the shift from qubit to q-string as the basic entity of quantum computation. At the same time it is important to see that ”No Cloning” is not necessarily an obstacle to the moving of data between one register and another in a quantum computer, a necessary to realizing practical quantum computation. Also, we show that there are a number of types of decoherence, and that understanding the difference between these allows us to build a quantum register whose processes are relatively stable in the external environment of a real physical machine.

In our future research, we plan to explore more deeply the properties of a q-string versus a string of qubits and to present a more formal proof of the difference between these two concepts. We shall study the notion of entanglement and how it relates to quantum parallelism. In particular, since decoherence is the major obstacle to achieving quantum parallelism, we would like to understand better the relationship between entanglement and decoherence. In addition, we want to make clear how ”No Cloning” affects the design and implementation of quantum memory and storage.

References

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