Sets

A set is a collection of objects. These objects can be anything, even sets themselves. An object \( x \) that is in a set \( S \) is called an element of that set. We write \( x \in S \).

Some special sets:

\( \emptyset \): The empty set; the set containing no objects at all

\( \mathbb{N} \): The set of all natural numbers 1, 2, 3, … (sometimes, we include 0 as a natural number)

\( \mathbb{Z} \): the set of all integers …, -3, -2, -1, 0, 1, 2, 3, …

\( \mathbb{Q} \): the set of all rational numbers (numbers that can be written as a ratio \( p/q \) with \( p \) and \( q \) integers)

\( \mathbb{R} \): the set of real numbers (points on the continuous number line; comprises the rational as well as irrational numbers)

When all elements of a set \( A \) are elements of set \( B \) as well, we say that \( A \) is a subset of \( B \). We write this as \( A \subseteq B \).

When \( A \) is a subset of \( B \), and there are elements in \( B \) that are not in \( A \), we say that \( A \) is a strict subset of \( B \). We write this as \( A \subset B \).

When two sets \( A \) and \( B \) have exactly the same elements, then the two sets are the same. We write \( A = B \).

Some Theorems:

For any sets \( A \), \( B \), and \( C \):

\[ A \subseteq A \]

\[ A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A \]

\[ \text{If } A \subseteq B \text{ and } B \subseteq C \text{ then } A \subseteq C \]

Formalizations in first-order logic:

\[ \forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)) \quad \text{Axiom of Extensionality} \]

\[ \forall x \forall y (x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y)) \quad \text{Definition subset} \]
Operations on Sets

With A and B sets, the following are sets as well:

- The union $A \cup B$, which is the set of all objects that are in A or in B (or both)
- The intersection $A \cap B$, which is the set of all objects that are in A as well as B
- The difference $A \setminus B$, which is the set of all objects that are in A, but not in B
- The powerset $P(A)$ which is the set of all subsets of A.

The Cartesian product $A \times B$, which is the set of all ordered pairs (2-tuples) $<a, b>$ where $a \in A$ and $b \in B$.

This generalizes to any number of sets, i.e. $A_1 \times A_2 \times \ldots \times A_n$ is the set of all $n$-tuples $<a_1, a_2, \ldots, a_n>$ where $a_1 \in A_1$, $a_2 \in A_2$, $\ldots$, and $a_n \in A_n$.

Some theorems:

$P(\emptyset) = \{\emptyset\}$

For any sets A, B, and C:

$A \cup B = B \cup A$ \hspace{1cm} \text{Commutation } \cup$

$A \cap B = B \cap A$ \hspace{1cm} \text{Commutation } \cap$

$A \cup (B \cup C) = A \cup (B \cup C)$ \hspace{1cm} \text{Association } \cup$

$A \cap (B \cap C) = A \cap (B \cap C)$ \hspace{1cm} \text{Association } \cap$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ \hspace{1cm} \text{Distribution } \cup \text{ over } \cap$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ \hspace{1cm} \text{Distribution } \cap \text{ over } \cup$

Formalization in FOL:

$\forall x \forall y \ (x \in \text{pow}(y) \iff x \subseteq y)$ \hspace{1cm} \text{Definition powerset}$

$\forall x \neg x \in e$ \hspace{1cm} \text{Definition empty set}$

$\forall x \forall y \forall z \ (z \in \text{union}(x,y) \iff (z \in x \lor z \in y))$ \hspace{1cm} \text{Definition union}$

$\forall x \forall y \forall z \ (z \in \text{int}(x,y) \iff (z \in x \land z \in y))$ \hspace{1cm} \text{Definition intersection}$

Exercise: formalize the difference operation (use dif(x,y))
Relations

A relation over sets $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \ldots \times A_n$

A relation over two sets, $A$ and $B$, is called a binary relation. For a binary relation $R$ we often write $aRb$ instead of $<a,b> \in R$. A binary relation $R$ over $A$ and $B$ is:

- **Left-Total** iff for all $a \in A$ there exists at least one $b \in B$ such that $aRb$
- **Right-Total** iff for all $b \in B$ there exists at least one $a \in A$ such that $aRb$
- **Right-Unique** iff for all $a \in A$ there exists at most one $b \in B$ such that $aRb$
- **Left-Unique** iff for all $b \in B$ there exists at most one $a \in A$ such that $aRb$

A binary relation that is both left-unique and right-unique is a **one-to-one relation**

A binary relation that is both left-total and right-total is a **correspondence**

$R$ is an **(endo-)relation on** $A$ if $R$ is a binary relation over $A$ and $A$. A relation $R$ on $A$ is:

- **Reflexive** iff $aRa$ for all $a \in A$
- **Non-Reflexive** iff not reflexive
- **Irreflexive** iff not $aRa$ for all $a \in A$
- **Symmetric** iff for all $a \in A$ and $b \in A$: if $aRb$ then $bRa$
- **Non-Symmetric** iff not symmetric
- **Asymmetric** iff for all $a \in A$ and $b \in A$: if $aRb$ then not $bRa$
- **Anti-symmetric** iff for all $a \in A$ and $b \in A$ where $a \neq b$: if $aRb$ then not $bRa$
  (or, what is the same thing: if $aRb$ and $bRa$ then $a = b$
- **Transitive** iff for all $a \in A, b \in A, \text{ and } c \in A$: if $aRb$ and $bRc$ then $aRc$
- **Non-Transitive** iff not transitive
- **Antitransitive** iff for all $a \in A, b \in A, \text{ and } c \in A$: if $aRb$ and $bRc$ then not $aRc$

Theorem: A relation $R$ is asymmetric if and only if $R$ is anti-symmetric and irreflexive

Exercise: Prove this (using non-formal, but still proper mathematical proof)
Formalization in FOL:

\[ \forall x \exists y \ R(x,y) \quad \text{R is Left-total} \]
\[ \forall y \exists x \ R(x,y) \quad \text{R is Right-total} \]
\[ \forall x \ \forall y \ \forall z \ ((R(x,y) \land R(x,z)) \rightarrow y = z) \quad \text{R is Right-unique} \]
\[ \forall x \ \forall y \ \forall z \ ((R(y,x) \land R(z,x)) \rightarrow y = z) \quad \text{R is Left-unique} \]
\[ \forall x \ R(x,x) \quad \text{R is reflexive} \]
\[ \forall x \ \neg R(x,x) \quad \text{R is irreflexive} \]
\[ \forall x \ \forall y \ (R(x,y) \rightarrow R(y,x)) \quad \text{R is symmetric} \]
\[ \forall x \ \forall y \ (R(x,y) \rightarrow \neg R(y,x)) \quad \text{R is asymmetric} \]
\[ \forall x \ \forall y \ ((R(x,y) \land R(y,x)) \rightarrow x = y) \quad \text{R is anti-symmetric} \]
\[ \forall x \ \forall y \ \forall z \ ((R(x,y) \land R(y,z)) \rightarrow R(x,z)) \quad \text{R is transitive} \]
\[ \forall x \ \forall y \ \forall z \ ((R(x,y) \land R(y,z)) \rightarrow \neg R(x,z)) \quad \text{R is intransitive} \]
\[ \forall x \ \forall y \ (R(x,y) \lor R(y,x)) \quad \text{R is total} \]
\[ \forall x \ \forall y \ (\neg x = y \rightarrow (R(x,y) \lor R(y,x))) \quad \text{R is connex} \]

Some special kinds of relations:

A reflexive, symmetric, and transitive relation is an \textit{equivalence relation}

A reflexive, anti-symmetric, and transitive relation is a \textit{partial order}

A total, anti-symmetric, and transitive relation is a \textit{total (or linear) order}

An irreflexive, anti-symmetric, and transitive relation is a \textit{strict partial order}

A connex, irreflexive, anti-symmetric, and transitive relation is a \textit{strict total order}

Examples:

\( \subseteq \) on sets is a partial (but non-strict) order, \( \subset \) is partial and strict
\( \leq \) on numbers is total (but non-strict) order, \( < \) is a strict total order

Exercise: Show that all total orders are partial orders.
Functions

A function $f : A \rightarrow B$ is a binary relation over $A$ and $B$ that is right-unique.

(For this reason, a binary relation that is right-unique is often called functional).

$A$ is called the domain, and $B$ the co-domain, of the function $f$ (note: many mathematicians use the word domain to refer to the set of objects for which a function-value is defined. We can call that the domain of definition. In the context of our course, however, it is more useful to regard the domain as the domain (or universe) of discourse.)

If there is an object $b$ such that $<a, b> \in f$, then $f(a)$ is defined as $b$, i.e. $f(a) = b$

If there is no object $b$ such that $<a, b> \in f$, then $f(a)$ is undefined

Frequently, the domain $A$ is a Cartesian product $A_1 \times A_2 \times \ldots \times A_n$. Accordingly, we will write $f : A_1 \times A_2 \times \ldots \times A_n \rightarrow B$ and $f(a_1, a_2, \ldots, a_n)$ instead of $f(<a_1, a_2, \ldots, a_n>)$

The range of a function $f : A \rightarrow B$ is the set of all $b \in B$ for which there exists some $a \in A$ such that $f(a) = b$

A function $f : A \rightarrow B$ is:

Total iff $f(a)$ is defined for all $a \in A$

(Thus, a total function is a right-unique, left-total binary relation)

Partial iff $f$ is not total

(Thus, a total function is a right-unique, but not left-total binary relation)

Surjective (or onto) iff for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$

(Thus, a surjective function is a right-unique, right-total binary relation. A right-total binary relation is sometimes called a surjective relation. Also, note that a function is surjective iff its range equals its co-domain)

Injective (or one-to-one) iff there do not exist $a_1 \in A$ and $a_2 \in A$ such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$

(Thus, an injective function is a right-unique, left-unique binary relation. A left-unique binary relation is sometimes called an injective relation.)

Bijective iff $f$ is surjective and injective

(So this is a right-unique, right-total, left-unique binary relation.)
A one–to-one correspondence between $A$ and $B$ iff $f$ is total and bijective

So this is a binary relation that has all 4 interesting properties: right-unique, right-total, left-unique, and left-total. Conceptually, this means that every element from $A$ corresponds to exactly one element in $B$, and vice versa. Indeed, the name makes sense: remember that a binary relation that is left-total and right-total is a correspondence, i.e. this is where there is at least one element from $B$ is associated for every element from $A$, and vice versa. Accordingly, a functional correspondence would be a function (right-unique) that is total (left-total) and onto (right-total). Add one-to-one (left-unique), and you not only have a functional one-to-one correspondence, but also a binary relation that is right-unique, right-total, left-unique, and left-total.

Somewhat more confusingly, a right-unique, right-total, left-unique, left-total binary relation is sometimes called a bijective relation. This seems confusing, because if for functions (right-unique), bijective simply means surjective (right-total) and injective (left-unique), where does the left-total property suddenly come from? Well, when mathematicians talk about functions, they often suppose them to be total (left-total!), unless explicitly specified to be partial. Indeed, when mathematicians talk about a bijective function, they often mean a one-to-one correspondence. Likewise, mathematicians often refer to a one-to-one correspondence as simply a correspondence.

For a one-to-one function function $f : A \to B$, we can define its inverse function $f^{-1} : B \to A$ as follows:

$f^{-1}(b) = a$ if $f(a) = b$
$f^{-1}(b) = \text{undefined}$ otherwise

Note that since $f$ is one-to-one, there can at most be one $a$ such that $f(a) = b$, so this function is well-defined.

For any two functions $f : A \to B$ and $g : B \to C$ we can define the composite function (or their composition) $g \circ f : A \to C$ as follows:

$g \circ f(a) = g(f(a))$ if $f(a)$ is defined
$g \circ f(a) = \text{undefined}$ otherwise

Exercise: Show that if $f$ is a one-to-one function, and $f^{-1}$ its inverse:

$f$ is total if and only if $f^{-1}$ is onto

$f$ is onto if and only if $f^{-1}$ is total